Journal of Statistical Physics, Vol. 27, No. 3, 1982

Decay of Correlations in Surface Models

R. Brandenberger[1] and C. E. Wayne[1,2]

Received June 29, 1981

A convergent low-temperature expansion for a variety of models of two-dimensional surfaces is presented. It yields existence of the thermodynamic limit for the pressure and correlation functions as well as analyticity in $z = e^{-\beta}$. In addition, the estimates give exponential decay of truncated correlations, which proves the existence of a gap in the spectrum of the transfer matrix below the ground state eigenvalue. Two particular examples included in the general framework are the solid-on-solid and discrete Gaussian models.

KEY WORDS: Surface models; decay of correlations; interfaces; algebraic formalism.

1. INTRODUCTION

Below we present an expansion for a class of models often used to study interfaces or crystal surfaces. This expansion is shown to converge at low temperatures and is used to study fluctuations out of the ground state of these models.

The models themselves are of the following sort. To each site in a two-dimensional square lattice we assign an integer which specifies the height of the interface at that site. The natural boundary conditions in this case consist of setting the heights identically equal to zero outside some finite volume. The energy of the interface is given by the Hamiltonian function[9]:

$$\mathcal{H}(\{h\}) = \sum_{\langle i,j\rangle} g(|h_i - h_j|)$$

The sum runs over nearest neighbor sites, $h_i \in \mathbb{Z}$ is the height of the interface at a given site, i, and $g(|\cdot|)$ is a sufficiently rapidly increasing

Supported in part by the National Science Foundation under grant No. PHY 79-16812.

[1] Department of Physics, Harvard University, Cambridge, Massachusetts 02138.
[2] NSF Predoctoral Fellow.

function. Specifically we assume

$$
\begin{gathered}
g(1) = 1 \\
\sum_{n=1}^{\infty} e^{-\beta g(n)} \leqslant e^{-\beta C}, \qquad \beta > 0, \qquad C \to 1 \text{ as } \beta \to \infty
\end{gathered} \tag{1}
$$

The low-temperature expansion is a generalized Peierls expansion. Since the energy of a height configuration depends only on the contours across which h changes its value, it is possible to use the algebraic framework which was originally discussed in Refs. 5 and 6 in the context of statistical mechanics and has recently been successfully applied to quantum field theory,[1,8] to derive cluster expansions for the pressure and correlation functions. Convergence of the expansion is proven using standard techniques and since the estimates are uniform in $\|\Lambda\|$ (the area of Λ) the existence of the thermodynamic limit follows. Our results are summarized in the following theorems ($z = e^{-\beta}$, $\beta = 1/kT$).

Theorem 1. There is some neighborhood, U, of the origin of the complex z plane and some constant c, independent of Λ and the choice of z within U such that

$$|Z_\Lambda| \leqslant e^{c\|\Lambda\|}$$

Furthermore the pressure

$$p(z) = \lim_{\|\Lambda\| \to \infty} \frac{1}{\|\Lambda\|} \log Z_\Lambda \tag{2}$$

exists and is analytic in z in U.

The second theorem deals with expectation values of functions f_π on the set of height configurations with support $\pi \subset \Lambda$ satisfying the following conditions:

$$
\begin{gathered}
f_\pi(\{h\}) = \prod_{i \in \pi} f_i(h_i) \\
|f_i(h_i)| \leqslant |h_i|^\alpha, \qquad \alpha \in \mathbb{R}^+
\end{gathered} \tag{3}
$$

Theorem 2. There exists a neighborhood, U, of $z = 0$ such that for any function f_π satisfying (3)

$$\left| \lim_{\|\Lambda\| \to \infty} \langle f_\pi(\{h\}) \rangle_\Lambda \right| < \infty$$

and is analytic as a function of z.

Assuming A_{π_1} and B_{π_2} satisfy (3), then with $d = \operatorname{dist}(\pi_1, \pi_2)$ there exist constants $0 < C_1, C_2 < \infty$ such that

$$|\langle A_{\pi_1} B_{\pi_2} \rangle - \langle A_{\pi_1} \rangle \langle B_{\pi_2} \rangle| \leqslant C_1 e^{-C_2 d}$$

In particular Theorem 2 establishes the existence of a gap $(e^{-C_2}, 1)$ in the spectrum of the transfer matrix (see Ref. 7) between the eigenvalue 1 corresponding to the ground state energy and the rest of the spectrum. We conclude the Introduction with several remarks:

(1) Two special cases included in the set of Hamiltonians allowed by (1) are the solid-on-solid (SOS) and discrete Gaussian models. In the SOS model $g(|h|) = |h|$.[8] The rigidity of the interface of this model at low temperatures is mentioned in Ref. 6. The discrete Gaussian model is defined by $g(|h|) = h^2$. This model is of particular interest because it is related, via a duality transformation, to the two-dimensional lattice Coulomb gas.[3,9]

(2) The systems our expansion is designed to handle are essentially two-dimensional spin systems in which the spin at any site may take on an infinite number of discrete values.

(3) These methods could presumably also be used to study the phase separated state of the three-dimensional Ising model, thereby reproducing the results established in Refs. 2 and 4 using a direct approach.

2. THE EXPANSION

A. Definition of Admissible and Compatible Cluster Configurations

The first step in our procedure is to set up a generalized Peierls expansion.

Definitions. (1) A contour Γ is a closed connected (possibly branching) set of dual lattice bonds.

(2) The cluster X associated with Γ is a $|\Gamma| + 1$ tuple $X = (\Gamma, h_1, h_2, \ldots, h_{|\Gamma|})$, $h_i \in Z \setminus \{0\}$ ($|\Gamma| =$ length of Γ).

(3) A cluster configuration is an unordered collection $\{X_1, \ldots, X_k\}$ of clusters.

In order to be able to rewrite Z in terms of a sum over cluster configurations we must first specify a way of associating a cluster configuration to a given height configuration. This will establish a bijective correspondence between the set of height configurations and a subset of all cluster configurations, the so-called admissible and compatible ones.

Given a simply connected region Λ with zero boundary conditions and a specified height configuration, the associated contours are defined to be the maximally connected components of the set of dual lattice bonds across which the height changes. Two different contours will thus be disjoint.

Definition. A set of clusters $\{X_1, \ldots, X_k\}$ is compatible if for all pairs (X_i, X_j) the associated contours satisfy $\Gamma_i \cap \Gamma_j = \emptyset$.

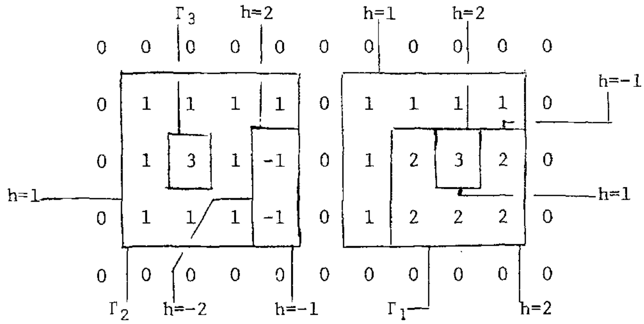

Fig. 1. Example of a cluster configuration for a given height configuration.

In order to define the clusters associated with the height configuration which gives contour Γ, it is convenient to first specify an ordering of the dual bonds: $b_1 < b_2$ if $x_{b_1} < x_{b_2}$ or if $x_{b_1} = x_{b_2}$ and $y_{b_1} < y_{b_2}$ where (x_b, y_b) are the coordinates of the midpoint of b. It is also necessary to specify a direction in which to measure the height differences: Given a bond b separating two connected components of $\operatorname{Int}\Gamma$ (R_1 and R_2) with n_1 (n_2) being the minimal number of links of Γ crossed by a line from R_1 (R_2) to $\Lambda\backslash\overline{\operatorname{Int}\Gamma}$, then the direction is from the region of smaller n to that of larger n. If $n_1 = n_2$ then the direction can be taken in direction of increasing order. This procedure is illustrated in Fig. 1. Finally the cluster associated with Γ can be defined to be

$$X = \left(\Gamma, h_{b_1}, h_{b_2}, \ldots, h_{b_{|\Gamma|}}\right), \qquad b_1 < b_2 < \cdots < b_{|\Gamma|}$$

(There is one height jump h_{b_i} associated with any bond b_i of Γ.)

Clearly the heights h_{b_i} arising from any height configuration must satisfy certain conditions. The jumps must be such that they add up to the same number for any line connecting $\partial\Lambda$ to a given point $p \in \operatorname{Int}\Gamma$.

Definition. A cluster configuration is said to be admissible if for any cluster Γ in the configuration and for any point $p \in \operatorname{Int}\Gamma$

$$\int_{\partial\Lambda}^{p} \Delta h \cdot ds \quad \text{is path independent}$$

$\int_c \Delta h \cdot ds$ is the line integral defined by adding up all heights jumps along c taking account of the above direction convention.

Corollary 1. There is a bijective correspondence between height configurations and admissible and compatible cluster configurations.

Corollary 2.

$$Z_\Lambda = \sum_{\substack{\{X_1, \ldots, X_k\} \\ \text{compatible} \\ \text{admissible} \\ \Gamma_i \subset \Lambda}} \prod_{s=1}^{k} \rho(X_s)$$

$$\rho(X_s) = \exp\left[-\beta \sum_{j=1}^{|\Gamma_s|} g(|h_{b_j}|)\right] \tag{4}$$

B. Algebraic Expansion for the Pressure

The basic idea underlying the algebraic method is to replace the summation over compatible cluster configurations in (4) by an unrestricted sum. (We shall assume from this point on that all clusters appearing are admissible unless it is explicitly stated otherwise.) This is accomplished by explicitly introducing the compatibility conditions as factors $[1 + A(X_i, X_j)]$ (we are using the notation of Refs. 1 and 8). Expanding the product of these factors in powers of A gives a set of terms each of which can be interpreted as a graph connecting certain of the vertices $X_1, \ldots, X_k$. The resulting cluster-graph expansion is a convenient starting point for convergence estimates since all clusters in a nonzero term must be connected (overlapping), enabling one to easily control the number of terms containing a given point p by the energy decay associated with the length of the contours.

The first step explicitly incorporates the compatibility conditions into the algebra.

$$A(X_i, X_j) \equiv \begin{cases} 0, & \Gamma_i \cap \Gamma_j = \emptyset \\ -1, & \text{otherwise} \end{cases} \tag{5}$$

The factor $1 + A(X_i, X_j)$ eliminates all configurations for which X_i and X_j are incompatible. Thus

$$Z_\Lambda = \sum_{\substack{\{X_1, \ldots, X_k\} \\ X_i \subset \Lambda}} \prod_{s_1 < s_2} \left[1 + A(X_{s_1}, X_{s_2})\right] \prod_{s=1}^{k} \rho(X_s)$$

Expanding the product of factors $1 + A(X_{s_1}, X_{s_2})$, and regarding $(X_1, \ldots, X_k)$ as a set of vertices and $A(X_{s_1}, X_{s_2})$ as a link connecting X_{s_1} and X_{s_2}, any term in the resulting sum is given by a graph Γ with at most one link l between any pair of vertices.

Let $G(X_1, \ldots, X_k)$ denote the set of all such graphs. Then

$$\begin{aligned} Z_\Lambda &= \sum_{\{X_1, \ldots, X_k\}} \prod_{s=1}^{k} \rho(X_s) \sum_{\Gamma \in G(X_1, \ldots, X_k)} \prod_{l \in \Gamma} A(l), \qquad X_i \subset \Lambda \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(X_1, \ldots, X_k)} \prod_{s=1}^{k} \rho(X_s) \sum_{\Gamma \in G(X_1, \ldots, X_k)} \prod_{l \in \Gamma} A(l) \end{aligned} \tag{6}$$

$(X_1, \ldots, X_k)$ denotes an *ordered* configuration of clusters.

In order to derive an expansion for the pressure, it is convenient to write Z_Λ as the exponential of what will turn out to be a sum over connected cluster-graph configurations (where connectivity this time refers to the graph). To accomplish this, we split every term in Z_Λ into the product of the contributions from each connected component (a single vertex is a possible connected component). Then we resum the series by summing first over the number n of connected components and then over the possible cluster-graph configurations in each component. The manipulations with infinite series are justified since the final series converge absolutely. $G_c(X_1, \ldots, X_k)$ is the set of all connected graphs of the set of vertices $X_1, \ldots, X_k$. V_i will denote the set of vertices of the ith connected component Γ_i of Γ. So

$$\begin{aligned} Z_\Lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(X_1, \ldots, X_k)} \sum_{\Gamma \in G(X_1, \ldots, X_k)} \prod_{i=1}^{n} \left[\prod_{l \in \Gamma_i} A(l) \prod_{j \in V_i} \rho(X_j) \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\substack{k_1, \ldots, k_n = 1 \\ \sum_1^n k_i = k}}^{k} \left(k! \Big/ \prod_1^n k_i! \right) \\ &\quad \times \prod_{i=1}^{n} \sum_{(X_1^i, \ldots, X_{k_i}^i)} \sum_{\Gamma \in G_c(X_1^i, \ldots, X_{k_i}^i)} \prod_{l \in \Gamma} A(l) \prod_1^{k_i} \rho(X_j^i) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{i=1}^{n} \left\{ \sum_{k_i=1}^{\infty} \frac{1}{k_i!} \sum_{(X_1^i, \ldots, X_{k_i}^i)} \sum_{\Gamma \in G_c(X_1^i, \ldots, X_{k_i}^i)} \left[\prod_{l \in \Gamma} A(l) \right] \prod_{j=1}^{k_i} \rho(X_j^i) \right\} \\ &= \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{(X_1, \ldots, X_k)} \sum_{\Gamma \in G_c(X_1, \ldots, X_k)} \left[\prod_{l \in \Gamma} A(l) \right] \prod_{j=1}^{k} \rho(X_j) \right\} \end{aligned} \tag{7}$$

The combinatorial factor $k!/\prod_1^n k_i!$ gives the number of ways of distributing k distinguishable objects into n piles with k_i elements in the ith pile.

Finally to obtain an expression for the pressure defined in (2), one remarks that all terms in $\log Z_\Lambda$ arise as translates of configurations containing the origin. There are $\|\Lambda\|$ possible translates. Any term, t, in $\log Z_\Lambda$ arises in $\|\bigcup_{s=1}^{k} X_s\|$ different ways as a translate. ($X_1, \ldots, X_k$ are the clusters of t.) Therefore up to boundary effects $p(\beta)$ will be given by the exponent of (7) restricted to terms containing the origin and divided by $\|\bigcup_{s=1}^{k} X_s\|$.

Lemma 1. $p(\beta)$ is given by the following cluster-graph expansion:

$$p(\beta) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{(X_1, \ldots, X_k) \\ 0 \in \bigcup_1^k X_s}} \sum_{\Gamma \in G_c(X_1, \ldots, X_k)} \prod_{l \in \Gamma} A(l) \prod_{j=1}^{k} \rho(X_j) \left(1 \Big/ \left\| \bigcup_{s=1}^{k} X_s \right\| \right) \tag{8}$$

Proof. Denote the right-hand side of (8) by $\hat{p}$. Define $S(\Lambda)$ by

$$\log Z_\Lambda \equiv \|\Lambda\| \hat{p} - S(\Lambda)$$

By the preceding remark all terms in $\log Z_\Lambda$ appear in $\|\Lambda\| \hat{p}$. Conversely, the only terms in $\|\Lambda\| \hat{p}$ which do not occur in $\log Z_\Lambda$ and thus form $S(\Lambda)$ are translates of terms t in $\hat{p}$ which cross $\partial\Lambda$. Since a cluster configuration crossing $\partial\Lambda$ can arise in $\|\Lambda \cap \bigcup_s X_s\|$ ways as a translate of a term in $\hat{p}$, it follows that

$$S(\Lambda) = \sum_{k=1} \frac{1}{k!} \sum_{\substack{(X_1, \ldots, X_k) \\ \partial\Lambda \cap \bigcup_s X_s \neq \emptyset}} \frac{\|\Lambda \cap \bigcup_s X_s\|}{\|\bigcup_s X_s\|} \times \sum_{\Gamma \in G_c(X_1, \ldots, X_k)} \left[\prod_{l \in \Gamma} A(l) \right] \prod_{j=1}^{k} \rho(X_j) \tag{9}$$

By the convergence estimates of Section 3,

$$|S(\Lambda)| \leqslant \text{const} |\partial\Lambda|$$

Hence

$$\lim_{\|\Lambda\| \to \infty} \frac{1}{\|\Lambda\|} \log Z_\Lambda = \hat{p} - \lim_{\|\Lambda\| \to \infty} O\left(\frac{|\partial\Lambda|}{\|\Lambda\|} \right) = \hat{p} \quad \blacksquare$$

C. Algebraic Expansion for the Expectation Values

In this section the algebraic methods are used to give a cluster-graph expansion for normalized expectation values of functions f_π satisfying the conditions (3). The new feature is that in order to isolate the contribution of f_π it is convenient to introduce two sets of clusters.

Definition. An X cluster (Y cluster) is a cluster with $\operatorname{Int} X \cap \pi \neq \emptyset$ ($\operatorname{Int} Y \cap \pi = \emptyset$).

For any cluster configuration f_π will depend only on the X clusters. There are new compatibility conditions for an admissible cluster configuration $\{X_1, \ldots, X_r; Y_1, \ldots, Y_k\}$, namely, $\operatorname{Int} X_i \cap \pi \neq \emptyset$ and $\operatorname{Int} Y_j \cap \pi = \emptyset$. Define

$$A(\pi, Y_j) = \begin{cases} -1 & \operatorname{Int} Y_j \cap \pi \neq \emptyset \\ 0 & \operatorname{Int} Y_j \cap \pi = \emptyset \end{cases}$$

Just as in the first step of part B all compatibility conditions on Y-type clusters are explicitly incorporated into the algebra by inserting factors $[1 + A(X_i, Y_j)]$, $0 \leqslant i \leqslant r$, $1 \leqslant j \leqslant k$; $[1 + A(Y_i, Y_j)]$, $1 \leqslant i < j \leqslant k$ (π is treated as a new cluster X_0). Thus the unnormalized expectation value of f_π can be written in the form

$$\begin{aligned} Z_\Lambda \langle f_\pi \rangle_\Lambda &= \sum_{\substack{\{X_1, \ldots, X_r; Y_1, \ldots, Y_k\} \\ \text{compatible}}} f_\pi(\{X_1, \ldots, X_r\}) \prod_{s=1}^{r} \rho(X_s) \prod_{j=1}^{k} \rho(Y_j) \\ &= \sum_{\substack{\{X_1, \ldots, X_r\} \\ \text{compatible} \\ \operatorname{Int} X_i \cap \pi \neq \emptyset}} f_\pi(\{X_1, \ldots, X_r\}) \prod_{s=1}^{r} \rho(X_s) \sum_{k=0}^{\infty} \frac{1}{k!} \\ &\quad \times \sum_{(Y_1, \ldots, Y_k)} \prod_{1}^{k} \rho(Y_j) \prod_{j_1 < j_2} \left[1 + A(Y_{j_1}, Y_{j_2})\right] \\ &\quad \times \prod_{\substack{0 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant k}} \left[1 + A(X_i, Y_j)\right] \end{aligned}$$

Expanding the final two products yields a sum of terms corresponding to all graphs in $G(X_0, \ldots, X_r; Y_1, \ldots, Y_k)$, the set of graphs of vertices $\{X_0, \ldots, X_r; Y_1, \ldots, Y_k\}$ containing no links between X vertices [since there are no $A(X_{i_1}, X_{i_2})$ factors] and with at most one link between any pair of vertices, i.e.,

$$\begin{aligned} Z_\Lambda \langle f_\pi \rangle_\Lambda = &\sum_{\substack{\{X_1, \ldots, X_r\} \\ \text{compatible} \\ \operatorname{Int} X_i \cap \pi \neq \emptyset}} f_\pi(\{X_1, \ldots, X_r\}) \prod_{s=1}^{r} \rho(X_s) \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(Y_1, \ldots, Y_k)} \\ &\times \prod_{1}^{k} \rho(Y_j) \sum_{\Gamma \in G(X_0, \ldots, X_r; Y_1, \ldots, Y_k)} \left[\prod_{l \in \Gamma} A(l)\right] \end{aligned}$$

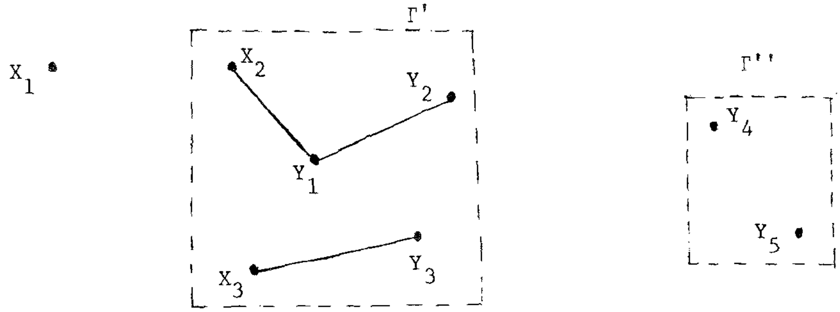

Fig. 2. Splitting of a graph $\Gamma \in G(X_1, X_2, X_3; Y_1, \ldots, Y_5)$.

The final step consists of separating all graphs into the part Γ' directly or indirectly connected to X clusters and the rest, Γ''. An example is shown in Fig. 2.

The resummation consists of summing separately over all Γ' and Γ'':

$$Z_\Lambda\langle f_\pi \rangle = \sum_{\substack{\{X_1, \ldots, X_r\} \\ \text{compatible} \\ \operatorname{Int} X_i \cap \pi \neq \emptyset}} f_\pi(\{X_1, \ldots, X_r\}) \prod_1^r \rho(X_s)$$

$$\times \sum_{k'=0}^{\infty} \frac{1}{k'!} \sum_{(Y_1', \ldots, Y_{k'}')} \prod_1^{k'} \rho(Y_j') \sum_{\Gamma' \in G_c(X_0, \ldots, X_r; Y_1', \ldots, Y_{k'}')} \left[\prod_{l \in \Gamma'} A(l) \right]$$

$$\times \sum_{k''=0}^{\infty} \frac{1}{k''!} \sum_{(Y_1'', \ldots, Y_{k''}'')} \prod_1^{k''} \rho(Y_j'') \sum_{\Gamma'' \in G(Y_1'', \ldots, Y_{k''}'')} \left[\prod_{l \in \Gamma''} A(l) \right]$$

$G_c(X_0, \ldots, X_r; Y_1', \ldots, Y_{k'}')$ is the set of graphs in $G(X_0, \ldots, X_r; Y_1', \ldots, Y_{k'}')$ connecting each Y_i' vertex (directly or indirectly) to an X vertex. The last line gives back Z_Λ. Thus:

$$\langle f_\pi \rangle = \sum_{\substack{\{X_1, \ldots, X_r\} \\ \text{compatible} \\ \operatorname{Int} X_i \cap \pi \neq \emptyset}} f_\pi(\{X_1, \ldots, X_r\}) \prod_1^s \rho(X_s)$$

$$\times \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(Y_1, \ldots, Y_k)} \prod_1^k [\rho(Y_j)]$$

$$\times \sum_{\Gamma \in G_c(X_0, \ldots, X_r; Y_1, \ldots, Y_k)} \left[\prod_{l \in \Gamma} A(l) \right] \tag{10}$$

3. CONVERGENCE ESTIMATES

A. Basic Convergence Lemma

Convergence of the expansions for both pressure and correlation functions will follow from the following basic lemma on the objects (see Ref. 8):

$$\Phi(X_1, \ldots, X_r; Y_1, \ldots, Y_k) \equiv \sum_{\Gamma \in G_c(X_1, \ldots, X_r; Y_1, \ldots, Y_k)} \prod_{l \in \Gamma} A(l) \prod_{s=1}^{k} \rho(Y_s)$$

which occur explicitly in (10) and implicitly in (8).

Lemma 2. There exists a neighborhood U of $z = 0$ such that for all $z \in U$

$$\sum_{\substack{(Y_1, \ldots, Y_k) \\ \sum_{s=1}^{k} |Y_s| = N}} |\Phi(X_1, \ldots, X_r; Y_1, \ldots, Y_k)| \leqslant k!\, |z|^{(k+N)/4} \exp\left(\sum_{i=1}^{r} |X_i| \right) \tag{11}$$

Proof. The proof proceeds by induction in $r + k$ and similarly to the basic convergence estimate in Ref. 6 is based on a Kirkwood–Salzburg-type equation which expresses $\Phi(X_1, \ldots, X_r; Y_1, \ldots, Y_k)$ in terms of Φ functions of less than $r + k$ vertices.

Given $\Gamma \in G_c$. Define

$\Omega = \{ s : l(Z_1, Y_s) \in \Gamma \}$; index set of Y clusters connected directly or indirectly to Z_1;

$\Gamma' = \{ l \in \Gamma : l = l(Z_j, Y_s)$ for $j > 1$ and $s \in \Omega$ or $l = l(Y_s, Y_{s'})$ for $s, s' \in \Omega \}$; set of graph links connecting $(Z_2, \ldots, Z_r)$ with clusters in Ω or connecting clusters in Ω among themselves;

$\Gamma'' = \Gamma \backslash \Gamma' \backslash \{ l(Z_1, Y_s) : s \in \Omega \}$.

With this notation,

$$\Phi(X_1, \ldots, X_r; Y_1, \ldots, Y_k) = \sum_{\Gamma \in G_c(X_1, \ldots, Y_k)} \prod_{s \in \Omega} \rho(Y_s) \prod_{s \in \Omega} A(X_1, Y_s) \times \prod_{l \in \Gamma'} A(l) \prod_{l \in \Gamma''} A(l) \prod_{s \notin \Omega} \rho(Y_s) \tag{12}$$

The Kirkwood–Salzburg equations follow by resumming (12): Fix Ω and sum over

(i) all possible graphs among vertices $(Y_s)_{s \notin \Omega}$ connected to $(X_j)_{j=2, \ldots, r}$ or $(Y_s)_{s \in \Omega}$, i.e., over the set $G_c(X_2, \ldots, X_r, (Y_s)_{s \in \Omega}; (Y_s)_{s \notin \Omega})$;

(ii) all graphs connecting $(Y_s)_{s\in\Omega}$ among themselves and to $(X_j)_{j=2,\dots,r}$ [this set is denoted by $G'(X_2,\dots,X_r;(Y_s)_{s\in\Omega})$].

Then only sum over Ω. Using in addition

$$\sum_{\Gamma'\in G'(X_2,\dots,X_r;(Y_s)_{s\in\Omega})}\prod_{l\in\Gamma'}A(l)=\prod_{\substack{j=2\\ s\in\Omega}}^{r}U(X_j,Y_s)\prod_{\substack{s_1<s_2\\ s_i\in\Omega}}U(Y_{s_1},Y_{s_2})$$

where $U(X_jY_s)=1+A(X_j,Y_s)$

Eq. (12) becomes the Kirkwood–Salzburg equation

$$\begin{aligned}\Phi(X_1,\dots,X_r;Y_1,\dots,Y_k)&\\ =\sum_{\Omega}\prod_{s\in\Omega}\rho(Y_s)\prod_{s\in\Omega}A(Z_1,Y_s)&\prod_{\substack{j=2\\ s\in\Omega}}^{r}U(X_j,Y_s)\prod_{\substack{s_1<s_2\\ s_i\in\Omega}}U(Y_{s_1},Y_{s_2})\\ \times\Phi(X_2,\dots,X_r,(Y_s)_{s\in\Omega};(Y_s)_{s\notin\Omega})&\end{aligned}\tag{13}$$

The left-hand side of (11) is estimated by inserting (13), resumming by first fixing the cardinality $|\Omega|$ of Ω and the total length $M\leqslant N$ of clusters in Ω, and summing over all cluster configurations compatible with $(|\Omega|,M)$. Using $|U(X_j,Y_s)|\leqslant 1$ and explicitly isolating the terms with $M=0$ and $M=N$ one obtains

$$\begin{aligned}&\sum_{\substack{(Y_1,\dots,Y_k)\\ \sum_{s=1}^k|Y_s|=N}}|\Phi(X_1,\dots,X_r;Y_1,\dots,Y_k)|\\ &\leqslant k!\sum_{|\Omega|=1}^{k-1}\frac{1}{|\Omega|!}\sum_{M=|\Omega|}^{N-1}\sum_{\substack{(Y'_1,\dots,Y'_{|\Omega|})\\ \sum|Y'_s|=M}}\prod_{s\in\Omega}\left[|A(X_1,Y'_s)|\,|\rho(Y'_s)|\right]\\ &\quad\times\frac{1}{(k-|\Omega|)!}\\ &\quad\times\sum_{\substack{(Y''_1,\dots,Y''_{k-|\Omega|})\\ \sum|Y''_s|=N-M}}|\Phi(X_2,\dots,X_r,Y'_1,\dots,Y'_{|\Omega|};Y''_1,\dots,Y''_{k-|\Omega|})|\\ &\quad+\sum_{\substack{(Y_1,\dots,Y_k)\\ \sum|Y_s|=N}}\prod_1^k\left[|A(X_1,Y_s)|\,|\rho(Y_s)|\right]\\ &\quad+\sum_{\substack{(Y_1,\dots,Y_k)\\ \sum|Y_s|=N}}|\Phi(X_2,\dots,X_r;Y_1,\dots,Y_k)|\end{aligned}$$

Using the induction hypothesis and Lemma 3 we obtain

$$\sum_{\substack{(Y_1, \ldots, Y_k) \\ \sum_{s=1}^k |Y_s| = N}} |\Phi(X_1, \ldots, X_r; Y_1, \ldots, Y_k)|$$

$$\leqslant k! \sum_{|\Omega|=1}^{k-1} \frac{1}{|\Omega|!} \sum_{M=|\Omega|}^{N-1} |X_1|^{|\Omega|} |z|^{3M/4} |z|^{(k-|\Omega|+N-M)/4} \exp\left(\sum_2^r |X_j|\right) e^M$$

$$+ |X_1|^k |z|^{3N/4} + k! \, |z|^{(k+N)/4} \exp\left(\sum_{i=2}^r |X_i|\right)$$

$$= k! \, |z|^{(k+N)/4} \exp\left(\sum_2^r |X_j|\right)$$

$$\times \left\{ 1 + \sum_{|\Omega|=1}^{k-1} \frac{|X_1|^{|\Omega|}}{|\Omega|!} |z|^{-|\Omega|/4} \sum_{M=|\Omega|}^{N-1} |z|^{M/2} e^M + \frac{|X_1|^k}{k!} |z|^{N/2-k/4} \right\}$$

For $|z|$ sufficiently small, the bracket is bounded from above by $e^{|X_1|}$, which completes the induction step. Since $\Phi(\varnothing; Y_1, \ldots, Y_k) \equiv 0$ and $\Phi(X_1, \ldots, X_r; \varnothing) \equiv 1$, Lemma 2 is proved. ■

Lemma 3. For all $z \in U$ (see Lemma 2)

$$\sum_{\substack{(Y_1, \ldots, Y_m) \\ \Gamma_s \cap \Gamma_{X_1} \neq \varnothing, \, s=1, \ldots, m \\ \sum |\Gamma_s| = M}} \prod_{s=1}^m |\rho(Y_s)| \leqslant |X_1|^m |z|^{3M/4}$$

Remark. Since $A(X_1, Y_s') \neq 0$ only if $\Gamma_{X_1} \cap \Gamma_{Y_s'} \neq \varnothing$, this estimate can indeed be used in Lemma 2.

Proof. In summing over all cluster heights it is possible to drop the admissibility conditions. Convergence is assured by the condition (1) on the Hamiltonian.

$$\sum_{\substack{(Y_1, \ldots, Y_m) \\ \Gamma_s \cap \Gamma_{X_1} \neq \varnothing \\ \sum |\Gamma_s| = M}} \prod_{s=1}^m |\rho(Y_s)| \leqslant \sum_{\substack{(\Gamma_1, \ldots, \Gamma_m) \\ \Gamma_s \cap \Gamma_{X_1} \neq \varnothing \\ \sum |\Gamma_s| = M}} \sum_{\substack{(h_1^1, \ldots, h_1^{|\Gamma_1|} \\ , \ldots, \\ h_m^1, \ldots, h_m^{|\Gamma_m|})}} \prod_{s=1}^m |z|^{\sum_{j=1}^{|\Gamma_s|} g(h_{sj})}$$

$$= \sum_{\substack{(\Gamma_1, \ldots, \Gamma_m) \\ \Gamma_s \cap \Gamma_{X_1} \neq \varnothing \\ \sum |\Gamma_s| = M}} \prod_{s=1}^m \prod_{j=1}^{|\Gamma_s|} \sum_{h \neq 0} |z|^{g(h)}$$

$$\leqslant \sum_{\substack{(\Gamma_1, \ldots, \Gamma_m) \\ \Gamma_s \cap \Gamma_{X_1} \neq \varnothing \\ \sum |\Gamma_s| = M}} |z|^{c_1 M}$$

c_1 can be chosen arbitrarily close to 1 by choosing U sufficiently small. The number of terms in this sum is bounded by $|\Gamma_{X_1}|^m e^{6M}$.[3] The final two factors may be incorporated by slightly decreasing c_1. For appropriate U the resulting c_1 will be greater than $3/4$. This completes the proof of the claim. ■

Note. If $X_1 = \pi$ then $|X_1| = \|\pi\|$ is to be used, the reason being that $A(\pi, Y_s) \neq 0$ iff $\operatorname{Int} Y_s \cap \pi \neq \emptyset$.[4]

B. Proof of the Main Results

Lemmas 2 and 3 are used to prove convergence of the expansion (8) for $p(z)$ and to show $|s(\Lambda)| \leqslant \text{const}|\partial\Lambda|$ as claimed in the proof of Lemma 1.

Lemma 4. For all $z \in U$ (see Lemma 2)

$$\left| \sum_{\substack{(Y_1, \ldots, Y_k) \\ \sum |Y_s| = N \\ p \in \bigcup_s Y_s}} \sum_{\Gamma \in G_c(Y_1, \ldots, Y_k)} \prod_{l \in \Gamma} A(l) \prod_{s=1}^{k} \rho(Y_s) \right| \leqslant k!\,|z|^{(k+N)/4} \tag{14}$$

Proof. Assume $p \in Y_1$ and compensate by multiplying by k. Then the left-hand side of (14) is bounded by

$$\begin{aligned}
&k \sum_{n=1}^{N-1} \sum_{\substack{(Y_1 \mid p \in Y_1) \\ |Y_1| = n}} \sum_{\substack{(Y_2, \ldots, Y_k) \\ \sum_2^k |Y_s| = N-n}} |\Phi(Y_1; Y_2, \ldots, Y_k)|\,|\rho(Y_1)| \\
&\leqslant k \sum_{n=1}^{N-1} \sum_{\substack{(Y_1 \mid p \in Y_1) \\ |Y_1| = n}} (k-1)!\,|z|^{(k-1+N-n)/4} e^n |z|^{\sum_{b \in \Gamma_1} g(h_b)} \\
&\leqslant k!\,|z|^{(k+N)/4} \sum_{n=1}^{N-1} |z|^{-(n+1)/4} e^n |z|^{3n/4} \\
&\leqslant k!\,|z|^{(k+N)/4}
\end{aligned}$$

[3] 2^M bounds the number of possibilities to split the total length M into m partial lengths $M_1, \ldots, M_m$. For a Γ_i with length M_i there are $|\Gamma_{X_1}|$ starting points. Since each Γ_i can be traced without passing a given bond more than twice, there are at most 4^{2M_i} choices of Γ_i with given starting point.

[4] Thus Y_s must encircle one point of π ($\|\pi\|$ possibilities). The number of possibilities for a path of length M_s to encircle a fixed point P is bounded by $\frac{1}{2} M_s 4^{2M_s}$, the first factor being the number of possible starting points of Γ_s along a path from P to infinity, the second arising as explained in footnote 3.

The second line follows by Lemma 2, the third by a combinatorial estimate analogous to the one in Lemma 3 and the final line by choosing U sufficiently small. ■

Corollary 3. The cluster-graph expansion (8) for $p(z)$ converges uniformly in Λ for all $z \in U$ and is analytic in U.

Proof. Uniform convergence follows immediately from Lemma 4. Since convergence is uniform in both Λ and in a height cutoff h_0 ($|h_b| \leqslant h_0$, for all b), since with these cutoffs the series is finite and hence trivially analytic and since $p(z)$ is uniformly bounded in z for $z \in U$, Vitali's theorem ensures analyticity of $p(z)$ in U. ■

Another immediate consequence of Lemma 4 is the following.

Corollary 4. For all $z \in U$

$$|S(\Lambda)| \leqslant \text{const}|\partial\Lambda| \quad ■$$

Since $\log Z_\Lambda = \|\Lambda\| p - S(\Lambda)$, Corollaries 3 and 4 also yield the first part of Theorem 1.

Lemma 2 is also the crucial input for the proof of the first half of Theorem 2.

Theorem 3. For all $z \in U$ and for any function f_π satisfying (3), the cluster-graph expansion (10) for $\langle f_\pi \rangle$ converges uniformly in Λ and is analytic in U.

Proof. Lemma 2 bounds the inner sum in (10). The sum over compatible X cluster configurations is estimated as in Lemma 3, the only additional complication being the presence of the f_π factor. If N is the total length of the Y clusters, then by Lemma 2

$$|\langle f_\pi \rangle| \leqslant e^{\|\pi\|} \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{\substack{(X_1, \ldots, X_r) \\ \text{compatible} \\ \text{Int}\, X_i \cap \pi \neq \varnothing}} \prod_{s=1}^{r} \rho(X_s) f_\pi(\{X_1, \ldots, X_r\})$$

$$\times \exp\left(\sum_1^r |X_s|\right)\left[\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{N=k}^{\infty} k!\, |z|^{(k+N)/4}\right]$$

U is chosen such that in addition to the previous requirements the square bracket is bounded by a constant c_2 for all $z \in U$. If $h_i = \max_{b \in \Gamma_i \cap \pi} |h_b|$, then by assumption (3)

$$f_\pi(\{X_1, \ldots, X_r\}) \leqslant \prod_{i=1}^{r} h_i^{\alpha\|\pi \cap \text{Int}\, \Gamma_i\|}$$

The sum over heights in one cluster X_i is bounded in the following steps: First it is assumed that the maximal height jump occurs for a particular bond b. To compensate it is necessary to multiply by $|\Gamma_i \cap \pi| \leqslant 4\|\pi\|^2$, then sum over $h_{b'}$ $(b' \neq b)$ as in Lemma 3 and over h_b using

$$\sum_{h_b \neq 0} |h_b|^{\alpha\|\pi \cap \operatorname{Int} \Gamma_i\|} |z|^{g(h_b)} \leqslant 2 \max_h \left(|h|^{\alpha\|\pi\|} |z|^{g(h)/2}\right) \sum_1^\infty |z|^{g(h_b)/2}$$

$$\leqslant M(\alpha, \|\pi\|, |z|) |z|^{c/2}$$

by (1) and defining $M(\alpha, \|\pi\|, |z|) \equiv 2 \max_h (|h|^{\alpha\|\pi\|} |z|^{g(h)/2})$ which is finite by (1). Since U is compact, it is possible to take M to be $|z|$ independent. Thus

$$|\langle f_\pi \rangle_\Lambda| \leqslant e^{\|\pi\|} \sum_{r=0}^\infty \frac{1}{r!} \sum_{m=4r}^\infty \sum_{\substack{(\Gamma_1, \ldots, \Gamma_r) \\ \text{compatible} \\ \operatorname{Int} \Gamma_i \cap \pi \neq \varnothing \\ \sum_{i=1}^r |\Gamma_i| = m}} \|\pi\|^{2r} M(\alpha, \|\pi\|)^r |z|^{cm/2}$$

As in the combinatorics of Lemma 3, the number of terms in the contour sum is bounded by $\|\pi\|^r (\text{const})^m$. Hence $\langle f_\pi \rangle_\Lambda$ converges uniformly in Λ with an absolute bound uniform in z for $z \in U$. As in the proof of Corollary 3, Vitali's theorem implies analyticity of $\langle f_\pi \rangle$ for $z \in U$. ∎

Exponential decay of truncated expectation values, the second part of Theorem 2, can be proved using the method of duplicate variables. The duplicate variables will be denoted by a prime, the expectation value in the product measure by $\langle\ \rangle^\sim$. For a function $f_{\pi \times \pi'}$ depending on both sets of variables with support π in the unprimed and π' in the primed variables, the cluster graph expansion (11) takes on the form

$$\langle f(\pi, \pi') \rangle^\sim = \sum_{\substack{\{X\} \, \{X'\} \\ \text{compatible}}} \sum f(\{X, X'\}) \prod_s \rho(X_s) \prod_{s'} \rho(X'_{s'}) \times \sum_k \frac{1}{k!}$$

$$\times \sum_{(Y_1, \ldots, Y_k)} \sum_{\Gamma \in G_c(X_0 \ldots X_r, Y_1 \ldots Y_k)} \prod_{l \in \Gamma} A(l) \prod_1^k \rho(Y_j) \sum_{i'} \frac{1}{k'!}$$

$$\times \sum_{(Y'_1 \ldots Y'_{k'})} \sum_{\Gamma' \in G_c(X'_0 \ldots X'_{r'}, Y'_1 \ldots Y'_{k'})} \prod_{l \in \Gamma'} A(l) \prod_1^{k'} \rho(Y'_j) \tag{16}$$

Truncated expectation values in the original model can be expressed as untruncated expectation values in the duplicate measure

$$\langle A_{\pi_1} B_{\pi_2} \rangle - \langle A_{\pi_1} \rangle \langle B_{\pi_2} \rangle = \tfrac{1}{2} \langle (A_{\pi_1} - A'_{\pi_1})(B_{\pi_2} - B'_{\pi_2}) \rangle^\sim \tag{17}$$

using the notation of Theorem 2. Now consider a term t in the expansion

(16) for the right-hand side of (17) with total cluster length $< 2d$. There will be a line l in $\mathbb{R}^2\backslash\pi$ separating π_1 from π_2 without intersecting t. The cluster configuration $\hat{t}$ obtained by interchanging original and duplicate clusters to the right of l has equal measure, but $B_{\pi_2} - B'_{\pi_2}$ has opposite sign, so in (17) t and $\hat{t}$ cancel. Thus $\langle (A_{\pi_1} - A'_{\pi_1})(B_{\pi_2} - B'_{\pi_2})\rangle^{\sim}$ contains only cluster configurations with total length $\geqslant 2d$. Thus one can pull out a factor $e^{-\beta 2d}$ from each term without affecting convergence of the remainder. This yields

Corollary 5. In the notation of Theorem 2

$$|\langle A_{\pi_1} B_{\pi_2}\rangle - \langle A_{\pi_1}\rangle\langle B_{\pi_2}\rangle| \leqslant c_1 e^{-c_2 d}$$

for sufficiently small temperatures.

This completes the proof of all the results stated in the Introduction.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge Professor A. Jaffe for encouraging this work and Professor J. Bricmont and Dr. J. Imbrie for suggesting several improvements in the manuscript.

REFERENCES

1. T. Balaban and K. Gawedzki, A Low Temperature Expansion for the Pseudoscalar Yukawa Model of Quantum Fields in Two Space-Time Dimensions. Harvard University preprint HUTMP 80/B87 (1980).
2. J. Bricmont, J. L. Lebowitz, and C. E. Pfister, *Commun. Math. Phys.* **69**:267 (1979).
3. S. T. Chui and J. D. Weeks, *Phys. Rev. B* **14**:4978 (1976).
4. R. L. Dobrushin, *Theory Prob. Appl.* **17**:582 (1972); *ibid.* **18**:253 (1973).
5. G. Gallavotti and S. Miracle-Sole, *Commun. Math. Phys.* **7**:274 (1968).
6. G. Gallavotti, A. Martin-Löf, and S. Miracle-Sole, in *Statistical Mechanics and Mathematical Problems*, Battelle Seattle 1971 Rencontres, A. Lenard, ed. (Springer, New York, 1973), p. 162 ff.
7. J. Glimm, A. Jaffe, and T. Spencer, in *Constructive Quantum Field Theory*, 1973 "Ettore Majorana" International School of Mathematical Physics, G. Velo and A. Wightman, eds. (Springer, New York, 1973), p. 132 ff.
8. J. Imbrie, Cluster Expansions and Mass Spectra for $P(\phi)_2$ Models Possessing Many Phases, Harvard University thesis (1980); *Commun. Math. Phys.* **82**:261(1981); *ibid* **82**:305(1981).
9. J. D. Weeks, in *Proceedings of the NATO Advanced Study Institute*, Geilo, Norway, 1979 Plenum Press, New York, 1980/ p. 293 ff.